ON THE ANALYTICAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS OF THE ELASTICITY THEORY FOR FINITE DOMAINS WITH THE ANGULAR POINTS OF A BOUNDARY AND THE CHANGING POINTS OF THE TYPE OF BOUNDARY CONDITIONS

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The paper focuses on the classical problem of the elasticity theory – the solution of the biharmonic equation in a semi-infinite strip (rectangle) and some conclusions that follow from the analysis of the exact solutions of the biharmonic problem. Interest in the solutions of boundary-value problems of the elasticity theory in canonical domains with the angular points of a boundary, particularly in a rectangle, never ceased, reacheing its peak in the years 1940–1980, primarily thanks to the Soviet school of mathematics and mechanics. Several thousands of papers were published throughout these years. The last review (2003) by V. V. Meleshko contains more than 700 references to the most important research in almost 200 years [1]. In those years 4 schools were established in the Soviet Union: the Leningrad school (P. F. Papkovich, A. I. Lurie, G. A. Grinberg, G. I. Dzhanelidze, V. K. Prokopov, A. V. Kostarev, S. G. Gurevich, B. M. Nuller and others), the Rostov-on-Don school (I. I. Vorovich, V. V. Kopasenko, V. E. Kovalchuk, Yu. A. Ustinov, V. I. Yudovich and others), the Kyiv school (V. T. Grinchenko, A. F. Ulitko, A. M. Gomilko, V. V. Meleshko and others), the Moscow school (M. I. Gusein-Zade, S. A. Lurie, V. V. Vasiliev, E. M. Zveryaev, V. I. Maly and others). After the 1980s there were no notable publications. The Western works were sketchy and weaker than the Soviet ones.

The heart of the problem is simple. Let us explain it with an example of the first fundamental boundary-value problem for the semi-infinite strip in the case of symmetric deformation: find the solution of the biharmonic equation in the semi-infinite strip $\{x \geq 0, |y| \leq 1\}$, that has free longitudinal sides, i.e.

$$\sigma_y(x,\pm 1) = \tau_{xy}(x,\pm 1) = 0, \tag{1}$$

and normal $\sigma_x(0,y) = \sigma(y)$ and tangential $\tau_{xy}(0,y) = \tau(y)$ stresses are set on the end face

$$\sigma_x(0,y) = \sigma(y),\ \tau_{xy}(0,y) = \tau(y). \tag{2}$$

Solving the problem by the method of separation of variables, we come to the problem of determining the coefficients a_k of expansions of the two functions $\sigma(y), \tau(y)$ set on the end face of the half-strip in series in two systems of the eigenfunctions of the boundary value problem – the so-called Fadle–Papkovich functions:

$$\sigma(y) = \sum_{k=1}^{\infty} a_k s_x(\lambda_k, y) + \overline{a_k} s_x(\overline{\lambda_k}, y),\ \tau(y) = \sum_{k=1}^{\infty} a_k t_{xy}(\lambda_k, y) + \overline{a_k} t_{xy}(\overline{\lambda_k}, y). \tag{3}$$

In the case of the symmetric deformation of the half-strip, the Fadle–Papkovich functions have the form:

$$\begin{aligned} s_x(\lambda_k, y) &= (1+\mu)\lambda_k\{(\sin\lambda_k - \lambda_k\cos\lambda_k)\cos\lambda_k y - \lambda_k y\sin\lambda_k\sin\lambda_k y\}, \\ t_{xy}(\lambda_k, y) &= (1+\mu)\lambda_k^2\{\cos\lambda_k\sin\lambda_k y - y\sin\lambda_k\cos\lambda_k y\}. \end{aligned} \tag{4}$$

μ is Poisson's ratio. The numbers λ_k are the set $\{\pm\lambda_k, \pm\overline{\lambda_k}\}_{k=1}^{\infty} = \Lambda$ of all the complex zeros of the entire function of the exponential type $L(\lambda) = \lambda + \sin\lambda\cos\lambda$. If the expansion coefficients a_k are found, the final solution of the boundary value problem in the half-strip will have the form (below only the expression for the stress σ_x is shown)

(5) $$\sigma_x(x,y) = \sum_{k=1}^{\infty} a_k s_x(\lambda_k, y) e^{\lambda_k x} + \overline{a_k} s_x(\overline{\lambda_k}, y) e^{\overline{\lambda_k} x}.$$

If the complex eigenvalues in obtaining the exact solutions of the boundary-value problem are put to the numbers $k\pi$ $(k = 1, 2, ...)$, the Fadle–Papkovich functions turn into usual trigonometric functions, and the solutions turn into the Filon–Ribiere solutions. The Fadle–Papkovich functions are much more difficult than the trigonometric ones: they are complex-valued and non-orthogonal. Moreover, they do not form a basis on the segment (on the half-strip's end face), and the expansions on them are not unique. Therefore, it is impossible to find the unknown coefficients of expansions, based on the classical concepts of the theory of the basis of functions. Numerous methods have been proposed for the determination of the coefficients of expansions in terms of the non-orthogonal sets of the Fadle–Papkovich functions. However, almost all of them, anyway, were brought to an approximate determination of the unknown coefficients from the infinite, irreducible system of algebraic equations. The basis of this method is the generalization of the classical notion of the basis of functions on the segment. If the classical basis on the segment can be considered to be a basis in the complex plane, the Fadle–Papkovich functions form a basis on the Riemann surface of the logarithm. The base for studying the basis properties of the Fadle–Papkovich functions is the theory of the quasi-entire functions of the exponential type and the Borel transform in this class of functions. The quasi-entire functions of the exponential type were first introduced by A. Pflüger in [2] in 1936. Based on Pflüger's results, it is possible to construct the systems of functions which are biorthogonal to the Fadle–Papkovich functions, and then exactly determine the expansion coefficients, thus constructing the exact solution of the problem. Such a solution was first published in [3].

In conclusion, we give formulas for stresses in the half-strip, when only the normal self-balanced stresses $\sigma_x(0,y) = \sigma(y)$ are set on its end face, and the tangential stresses are equal to zero

(6) $$\sigma_x(x,y) = \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{\sigma_k \frac{s_x(\lambda_k, y)}{M_k} \frac{\mathrm{Im}(-\overline{\lambda_k} e^{\lambda_k x})}{\mathrm{Im}(\lambda_k)}\right\},$$
$$\sigma_k = \int_{-1}^{1} \sigma(y) \frac{\cos\lambda_k y}{2(1+\mu)\lambda_k \sin\lambda_k} dy, \; M_k = L'(\lambda_k) = \cos^2\lambda_k.$$

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