ON THE VARIATIONAL FORMULATION OF THE EXTENDED HIGH-ORDER SHELL THEORY OF I. N.VEKUA TYPE

S. I. Zhavoronok[1]
[1]*Institute of Applied Mechanics of Russian Academy of Sciences, Moscow, Russia*

1. Introduction

The dimensional reduction [1] remains one of most powerful methods of construction of the hierarchy of refined shell models [2, 3, 4]. Here the further development of high-order shell theories [5, 6] is proposed. The main improvement consists in the satisfaction of the boundary conditions on the shell faces. The surface density of Lagrangian as well as the constraint equations defined on the shell coordinate surface are derived from the Lagrangian volumetric density and the boundary conditions on the faces using the biorthogonal expansion technique. The dynamic equations are formulated as Lagrange equations of the second kind on the groundwork of the Lagrange multipliers method. The new generalized forces accounting the effect of the constraints are introduced.

2. Generalized Lagrange's equations of the II'nd kind for constrained continuous systems

Let us consider a continuous mechanical system on $S \subset \mathbb{R}^n$. Let the system be defined within the configuration space Ω, with the field variables $q_I(M,t)$, $I = 1 \dots N$, $M \in S$, $t \in \mathbb{R}_+ \cup 0$, the spatial $L_S(q_I, \dot{q}_I, L_J[q_I])$ and hypersurface $L_\Gamma(q_I)$ Lagrangian densities [5, 6], and the linear constraints $f^Q(q_I, \mathrm{C}_P[q_I]) = 0$, $Q = 1 \dots M_F$, where $\mathbb{L}_J$, $J = 1 \dots M_J$ and C_P, $P = 1 \dots M_C$ are linear operators [7]. Let also the spatial $(u,v)_S$ and hypersurface $(u,v)_\Gamma$ scalar products be defined by the appropriate integrals [5]. Thus, we can derive the Lagrange equations of the 2'nd kind [8] and the natural boundary conditions for the considered system in the following form [7]:

$$-\frac{\partial}{\partial t}\left(\frac{\partial L_S}{\partial \dot{q}_I}\right) + \frac{\partial L_S}{\partial q_I} + \lambda_Q \frac{\partial f^Q}{\partial q_I} + \mathrm{L}^*_J\left[\frac{\partial L_S}{\partial\left(\mathrm{L}_J\left[q_I\right]\right)}\right] + \mathrm{C}^*_P\left[\lambda_Q \frac{\partial f^Q}{\partial\left(\mathrm{C}_P\left[q_I\right]\right)}\right] = 0; \tag{1}$$

$$\left\{\frac{\partial L_\Gamma}{\partial q_I} + \mathrm{B}^L_J\left[\frac{\partial L_S}{\partial\left(\mathrm{L}_J\left[q_I\right]\right)}\right] + \mathrm{B}^C_P\left[\lambda_Q \frac{\partial f^Q}{\partial\left(\mathrm{C}_P\left[q_I\right]\right)}\right]\right\} \delta q_I\Big|_{\partial S} = 0. \tag{2}$$

Here λ_Q denote the Lagrange multiplier; L^*_J, C^*_P are the operators being adjoined to L_J, C_P with respect to the scalar products, and B^L_J, B^C_P are appropriated boundary operators ([5, 6] and [7]).

3. Formulation of the shell model as a two-dimensional constrained mechanical system

Let us consider an elastic shell $V \subset \mathbb{R}^3$ with smooth faces $S_\pm$. Let $\mathbf{s}$ be a stress tensor, $\mathbf{u}$ be a displacement vector, $\mathbf{C}$ be an elastic constants tensor. The boundary conditions on $S_\pm$ can be represented as $\mathbf{s}|_\pm \cdot \mathbf{n}_\pm \equiv [\mathbf{C} : (\nabla \otimes \mathbf{u})]|_\pm \cdot \mathbf{n}_\pm = \mathbf{q}_\pm$ where $\mathbf{q}_\pm$ is the resultant force vector. The 2D shell model is defined on the two-dimensional manifold S corresponding to the base surface. We can "shift" the boundary conditions from $S_\pm$ to S representing the vector $\mathbf{u}$ in the basis $\mathbf{r}^\alpha, \mathbf{n}$ defined on the tangent fibration $T_M S$ [7]. Now the dimensional reduction can be applied; let us introduce the biorthogonal expansions [5, 6] for the displacement vector, $\mathbf{u} = (u^{(k)}_\alpha \mathbf{r}^\alpha + u^{(k)}_3 \mathbf{n})\mathrm{p}_{(k)}(\zeta)$, $\alpha = 1, 2$, $k = 0 \dots N$, $\zeta \in [-1, 1]$ is the normal coordinate. Therefore the Lagrangian surface density on S can be formulated [5, 6], and the boundary conditions on $S_\pm$ become the constraints defined on S [7, 9]:

$$\begin{gathered}\bar{\bar{C}}^{i\gamma\beta}_{\pm(k)}\left(\bar{\nabla}_\delta u^{(k)}_\gamma + H^{(\cdot k)}_{\delta(m\cdot)} u^{(m)}_\gamma - b_{\gamma\delta} u^{(k)}_3\right) + \bar{\bar{C}}^{i\gamma 3}_{\pm(k)}\left(\bar{\nabla}_\delta u^{(k)}_3 + H^{(\cdot k)}_{\delta(m\cdot)} u^{(m)}_3 + b^\gamma_\delta u^{(k)}_\gamma\right) + \\ + \bar{C}^{i3j}_{\pm(k)} h^{-1} D^{(\cdot k)}_{(m\cdot)} u^{(m)}_j - \mu_\pm q^i_\pm = 0, \quad i, j = 1 \dots 3.\end{gathered} \tag{3}$$

Here the "surface" values $\bar{\bar{C}}^{ij\delta}_{\pm(k)} = \bar{\bar{C}}^{i3j\delta}_{\pm} \mathrm{p}_{(k)}(\pm 1) + h^{\pm}_{\beta} \bar{\bar{C}}^{i\beta j\delta}_{\pm} \mathrm{p}_{(k)}(\pm 1)$; $\bar{\bar{C}}^{ijpq}_{\pm} = \bar{\bar{C}}^{\,ijpq}|_{\zeta=\pm 1}$ [7] of the generalized elastic constants $\bar{\bar{C}}^{ijpq}$ [6, 7] are introduced. Thus, The Lagrange equations (4, 5) and their natural boundary conditions (6) can be represented in the following formulation [7]:

$$\rho^{(m)}_{(k)} \ddot{u}^{\alpha}_{(m)} = \bar{\nabla}_{\beta} \tilde{s}^{\alpha\beta}_{(k)} - H^{(\cdot m)}_{\beta(k\cdot)} \tilde{s}^{\alpha\beta}_{(m)} - b^{\alpha}_{\beta} \tilde{s}^{3\beta}_{(k)} - h^{-1} D^{(\cdot m)}_{(k\cdot)} \tilde{s}^{\alpha 3}_{(m)} + \tilde{P}^{\alpha}_{(k)}; \tag{4}$$

$$\rho^{(m)}_{(k)} \ddot{u}^{3}_{(m)} = \bar{\nabla}_{\beta} \tilde{s}^{3\beta}_{(k)} - H^{(\cdot m)}_{\beta(k\cdot)} \tilde{s}^{3\beta}_{(m)} + b_{\alpha\beta} \tilde{s}^{\alpha\beta}_{(k)} - h^{-1} D^{(\cdot m)}_{(k\cdot)} \tilde{s}^{33}_{(m)} + \tilde{P}^{3}_{(k)}; \tag{5}$$

$$\left(\tilde{s}^{i\,\beta}_{(k)} \nu_{\beta} - q^{i}_{B(k)} \right) \delta u^{(k)}_{i} \Big|_{M_0 \in \Gamma} = 0; \quad i = 1, 2, 3; \quad \beta = 1, 2. \tag{6}$$

where $H^{(\cdot m)}_{\beta(k\cdot)}$ and $D^{(\cdot m)}_{(k\cdot)}$ are linear operators [5]. The shell model consists in (4, 5, 6), the kinematic equations and the initial conditions (see [5, 6]), the constraint equations (3), and the constitutive equations represented as follows [7]:

$$\tilde{s}^{ij}_{(k)} = \bar{\bar{C}}^{ij\gamma\delta}_{(km)} \bar{\nabla}_{\delta} u^{(m)}_{\gamma} + \bar{\bar{C}}^{ij3\delta}_{(km)} \bar{\nabla}_{\delta} u^{(m)}_{3} + \bar{C}^{ij\gamma}_{(km)} u^{(m)}_{\gamma} + \bar{C}^{ij3}_{(km)} u^{(m)}_{3} + \lambda^{\pm}_{\varepsilon} \bar{\bar{C}}^{\varepsilon ij}_{\pm(k)} + \lambda^{\pm}_{3} \bar{\bar{C}}^{3ij}_{\pm(k)}; \tag{7}$$

the new forces $\tilde{s}^{ij}_{(k)}$ account the boundary conditions on $S_{\pm}$ by means of the Lagrange multipliers $\lambda^{\pm}_{i}$.

4. Conclusions

Now the 2D model allows one to obtain solutions satisfying the boundary conditions on the faces of a shell as well as to estimate the shell stiffness properly on the basis of low-order theories. The "elementary" shell theory of N'th order [5, 6] follows from the "extended" one [7, 9] if the constraints (3) are neglected. The effect of the constraints is analyzed using the solutions of some test problems of plate and shell dynamics [9]. To formulate the presented model in terms of first-order partial differential equations of Hamilton type [8] a generalized Nambu formulation [10] is efficient.

Author is supported by the Russian Foundation for Basic Researches (grant No.16-01-00751-a).

5. References

[1] I. N. Vekua (1985). *Shell Theory: General Methods of Construction*, Pitman Advanced Publ. Program, Boston.

[2] C. Schwab, S. Wright (1995). Boundary layers in hierarchical beam and plate models, *J. Elast.*, **38**, 1-40.

[3] D. Gordeziani, M. Avalishvili, and G. Avalishvili (2006). Hierarchical models for elastic shells in curvilinear coordinates, *Comput. Math. Appl.* **51**, 1789-1808.

[4] V. V. Zozulya (2015). A higher order theory for shells, plates and rods, *I.J.Mech.Sci.*, **103**, 40-54.

[5] S. I. Zhavoronok (2014). A Vekua-type linear theory of thick elastic shells, *ZAMM*, **94**, 164-184.

[6] S. I. Zhavoronok (2014). Variational formulations of Vekua-type shell theories and some their applications. In: *Shell Structures: Theory and Applications.* Vol. 3. CRC Press / Balkema, Taylor & Francis Gr., 341-344.

[7] S. I. Zhavoronok (2015). On the variational formulation of the extended thick anisotropic shells theory of I. N. Vekua type, *Procedia Engineering*, **111**, 888-895.

[8] N. A. Kilchevskiy, G. A. Kilchinskaia, N. E. Tkachenko (1979), *Fundamentals of the Analytical Mechanics of Continuous Systems*, Naukova Dumka, Kiev, 1979.

[9] O. V. Egorova, S. I. Zhavoronok, and A. S. Kurbatov (2015). An application of various n-th order shell theories to normal waves propagation problems, *PNRPU Mechanics Bulletin*, No.2, 36-59.

[10] C. C. Lassig and G. C. Joshi, Constrained systems described by Nambu mechanics (1997). *Letters in Mathematical Physics*, **41**, 59-63.