# VERY HIGH-ORDER ELEMENTS IN THERMAL AND MECHANICAL PROBLEMS

***J. Jaśkowiec***
*Cracow University of Technology,*
*Institute for Computational Civil Engineering, Cracow, Poland*

## 1. Introduction

In this paper the very high-order elements are considered which are understood as elements for which the approximation order exceeds 10. In scientific literature a method is called as high-order when polynomial order $p$ in single element is greater than 2, but on the other hand the approximation order $p$ does not exceeds 10 e.g. [1]. Usually, using $p > 10$ leads to numerical instabilities due to severe truncation errors. This happens due to fact that the approximations are usually based on Lagrange polynomials. When other kinds of polynomials in finite elements are used then we can go far beyond $p = 10$. Instead the Chebyshev or Legendre polynomials can be used as basis functions. Approximation based on Lagrange polynomials is continuous in the finite element mesh. In a case of Chebyshev or Legendre polynomials the approximation continuity is no longer true, what is a typical situation in discontinuity Galerkin (DG) method [5].

This work is direct extension of the author's papers [4, 3, 2]. In this work the great computational effectiveness is shown for very high-order approach. There are shown examples solved by low-order and very high-order elements. In a case of low-order elements the mesh has to be quite dense. On the other hand, a rough mesh is enough for very high-order elements. In consequence, the approximate solution based on low-order element can be achieved even after few hours. When high-order elements are used it is a matter of seconds to solve the same problem with similar solution quality. This paper deals also with $hp$ mesh refinement based on Zienkiewicz-Zhu (ZZ) error estimator. The ZZ method is adjusted to DGFD method so that the error indicator is calculated for each element which is afterwards used for the mesh refinement. The $h$-type refinement is connected with the division of elements into smaller ones. In DG methods the finite element do not have to be conforming, so the $h$ mesh refinement is much easier in comparison to refinement in finite element method. The $p$-type refinement is straightforward, since raising approximation order of one element we do not need change anything in the neighbouring elements. In the approach presented in this work the $p$ refinement is practically unlimited since $p$ can goes much over 20. In result in one mesh the element order may vary from $p = 1$ to $p = 50$.

This paper is illustrated with a series of examples presenting the properties and possibilities of the very high-order DGFD method. The examples show also the effect of $hp$ refinement in the method for both heat transport and plane stress structure problems in 2D domains.

## 2. DGFD method description

In the paper the DGFD method is formulated for 2D scalar elliptic problem (heat transport) as well as for vector elliptic problem (structure in the plane stress state). The final formulations of the DGFD method for the both kinds of problem are different, but the main idea lying behind is the same. In the method the heat flux or the stress tensor have to evaluated on the mesh skeleton which are calculated by a fourth order finite difference rule that uses the approximations on two adjacent elements. It leads to compatibility conditions that enforces continuity of the final solution. Quite similar approach is applied on the outer boundary to enforce Dirichlet boundary conditions.

The DGFD method constructed in such a way is stable, robust and gives correct results. The nice property of DGFD method is the fact that quite arbitrary finite elements can be applied in the

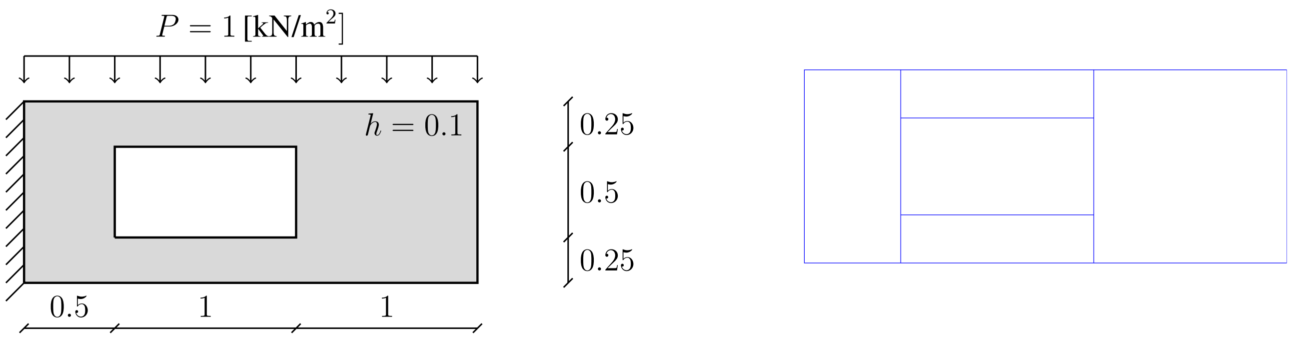


**Figure 1.** Cantilever beam with rectangular hole: the geometry and mesh. Dimensions in [m]

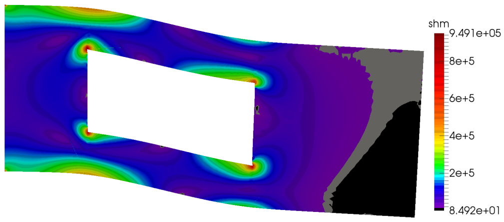


**Figure 2.** Von Mises stress in the deformed cantilever beam with hole calculated by very high-order elements

mesh, e.g. triangle, quadrangle, pentagon or hexagon, and so on. They can be convex as well as non-convex or even consists of two separate parts. The DGFD method allow to use different kinds of approximation basis functions, so they can be Chebyshev or Legendre basis functions. Their orthogonality properties can be exploited to reduce truncations errors for very high-order polynomials.

## 3. Examples

In this paper various examples are prepared illustrating various aspects of the very high-order DGFD method. Here a specific example is presented that shows the potential of the method. The cantilever beam in plane stress state with the rectangular hole inside subjected to outer load is considered, see Fig. 1. In this example the stresses are expected to concentrate on the rectangular hole vertices. The mesh consists only of four non-conforming rectangular finite elements each of order $p = 60$. The results are shown in the form of von Mises stress in the deformed cantilever in Fig. 2.

## 4. References

[1] W. Cecot and M. Oleksy. High order FEM for multigrid homogenization. *Comput Math Appl*, 70(7):1391 – 1400, 2015. High-Order Finite Element and Isogeometric Methods.

[2] J. Jaśkowiec. Application of discontinuous Galerkin method to mechanical 2d problem with arbitrary polygonal and very high-order finite elements. *Comput Struct*, :submitted in April, 2016.

[3] J. Jaśkowiec. Very–high order discontinuous Galerkin method. *Comput Method Appl M*, :submitted in May, 2016.

[4] J. Jaśkowiec, P. Pluciński, and A. Stankiewicz. Discontinuous Galerkin method with arbitrary polygonal finite elements. *Finite Elem Anal Des*, :in revision, 2016.

[5] B. Rivière. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations*. Society for Industrial and Applied Mathematics, 2008.