INVERSE HOMOGENIZATION IN ISOTROPIC MATERIAL DESIGN

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1. Introduction

Recent rapid development of the 3D printing technology makes it possible to bridge a gap between the theory of structural topology optimization and practical application of optimal designs. However, this application is not straightforward if the microstructural nature of 3D-printouts is not explicitly taken into account in the formulation of the optimization problem.

In linear elasticity, to which we restrict our research, it is justified to split that task into: (a) determining optimal effective properties of a composite in the design domain; (b) identifying constitutive parameters of basic materials ("the ink" for 3D printer) and their local microstructural arrangement (the printing pattern). Full generality is achieved by assuming that subtasks (a) and (b) are respectively performed by Free Material Design (FMD) and inverse homogenization. In this paper we limit our considerations in (a) to Isotropic Material Design (IMD). Details on homogenization theory, FMD, IMD, and comprehensive list of references, may be obtained from [1, 5, 6].

2. Notation

Fix $\boldsymbol{C}_*$ for the result of the FMD. Here $\boldsymbol{C}_* : \Omega \to \mathbb{E}^s_4$, $\Omega \subset \mathbb{R}^N$, $N = 2$ or 3, and $\mathbb{E}^s_4$ stands for the space of Hooke's tensors on $\mathbb{R}^N$. Next, refer to the homogenization theory for the notion of G-closure $G(\mathcal{D})$ of a given set $\mathcal{D} = \{\boldsymbol{D}_K\}_{1\le K\le k}$, $k \in \mathbb{N}$, whose elements are Hooke's tensors. Loosely speaking, $G(\mathcal{D})$ is obtained by considering all microstructural compositions of materials from $\mathcal{D}$ in all volume fractions $\{\theta_K\} : \Omega \to [0,1]$, restricted by $\theta_1(x) + \ldots + \theta_K(x) = 1$, $x \in \Omega$.

Inverse homogenization naturally requires $\boldsymbol{C}_*(x) \subset G(\mathcal{D})$ for all $x \in \Omega$; the goal is to determine the set $\mathcal{D}$; volume fractions $\{\theta_K\}_{1\le K\le k}$, and local microstructures at each $x \in \Omega$. The important point to note here is that inverse homogenization does not give unique results. In this paper, we exclude the non-uniqueness of local microstructures from the discussion; the Reader is referred to [2, 3, 7] for various details of this topic. For brevity of notation, in the sequel we will write $Hom^{-1}(\boldsymbol{C}_*)$ to denote the results of inverse homogenization performed on the field $\boldsymbol{C}_*$.

3. Inverse homogenization in IMD

In this section we consider several issues of $Hom^{-1}(\boldsymbol{C}_*)$ where $\boldsymbol{C}_* = k_*\,\boldsymbol{\Lambda}_1 + \mu_*\,\boldsymbol{\Lambda}_2$ is the result of IMD. Here $k_* = k_*(x)$ and $\mu_* = \mu_*(x)$, $x \in \Omega$, respectively denote the fields of optimal Kelvin and Kirchhoff moduli; $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ stand for the projectors of tensor $\boldsymbol{C}_*$ on the spherical and deviatoric subspaces of $\mathbb{E}^s_4$. We assume $\boldsymbol{C}_* \subset G_{iso}(\mathcal{D})$, where $G_{iso}(\mathcal{D})$ denotes the subset of isotropic effective tensors in $G(\mathcal{D})$.

Set $\mathcal{D} = \{\boldsymbol{0}, \boldsymbol{D}\}$ where $\boldsymbol{0}$ represents void and $\boldsymbol{D}$ stands for a given isotropic tensor with moduli k and μ. Volume fractions of $\boldsymbol{D}$ and void are respectively denoted by θ and $1-\theta$. It follows from [4] that k_{eff}, μ_{eff} representing $\boldsymbol{D}_{eff} \in G_{iso}(\mathcal{D})$ are bounded by $0 \le k_{eff} \le k_{HS}$, $0 \le \mu_{eff} \le \mu_{HS}$ where k_{HS}, μ_{HS} stand for the celebrated Hashin-Shtrikman bounds (HS for short). They read

$$
\begin{aligned}
k_{HS} &= \frac{\theta\, k k_0}{(1-\theta)k + k_0}, \quad k_0 = \frac{2(N-1)}{N}\,k, \\
\mu_{HS} &= \frac{\theta\, \mu\mu_0}{(1-\theta)\mu + \mu_0}, \quad \mu_0 = \frac{2(N^2-N-2)\mu + N^2 k}{2N(k+\mu)}\,\mu.
\end{aligned}
\tag{1}
$$

Note that it is impossible to have $k_* = k_{HS}$ and $\mu_* = \mu_{HS}$ everywhere in the design domain Ω. Indeed, for $N = 2$ these requirements force

$$(2) \qquad \frac{1}{k_*(x)} - \frac{1}{2\mu_*(x)} = \frac{1}{k} - \frac{1}{2\mu}, \qquad \theta(x) = \frac{k_*(x)}{k}\frac{k+\mu}{k_*(x)+\mu}, \qquad \theta(x) \le 1.$$

Moduli k and μ are constant and fields k_*, μ_* vary independently hence the first equality in (2) cannot be reached in the whole Ω. For $N = 3$ similar arguments apply.

Relaxing $\mu_* = \mu_{HS}$ to $0 \le \mu_* \le \mu_{HS}$ and assuming $N = 2$ gives

$$(3) \qquad \frac{1}{k_*(x)} - \frac{1}{2\mu_*(x)} \ge \frac{1}{k} - \frac{1}{2\mu}.$$

Inequalities in (2) and (3) have to be fulfilled for all $x \in \Omega$ which leads to

$$(4) \qquad \frac{1}{k} - \frac{1}{2\mu} \le \min_{x\in\Omega}\left\{\frac{1}{k_*(x)} - \frac{1}{2\mu}, \frac{1}{k_*(x)} - \frac{1}{2\mu_*(x)}\right\}.$$

By (4) one may adjust k and μ to obtain $\theta(x) \le 1$, $0 \le \mu_* \le \mu_{HS}$ and $k_*(x) = k_{HS}(x)$, $x \in \Omega$. The latter means that it is possible to find a microstructure with maximal bulk modulus in the whole design domain.

Discussion of the case $N = 3$ is omitted for the reason of space.

4. Concluding remarks

From this paper we conclude that matching the IMD-optimal fields k_*, μ_* with HS bounds k_{HS}, μ_{HS} everywhere in the design domain is possible only if the set $G(\mathcal{D})$ is determined by sufficient number of controls. This may be achieved by dropping the isotropy assumption on $\boldsymbol{D}$ or by enlarging the number of isotropic phases in $\mathcal{D}$. Universality of $Hom^{-1}(\boldsymbol{C}_*)$ allows for generalizing the discussion of this paper into FMD problems related to other types of Hooke's tensor symmetry. It is also possible to consider engineering theories formulated within the framework of linearized elasticity.

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6. References

[1] G. Allaire (2002). *Shape Optimization by the Homogenization Method*, Springer-Verlag, New York.

[2] C. Barbarosie (2003). Shape optimization of periodic structures, *Comp. Mech.*, **30**, 235-246.

[3] A. Cherkaev and G. Dzierżanowski (2013). Three-phase plane composites of minimal stress energy: High-porosity structures, *Int. J. Solids Structures*, **50**, 4145-4160.

[4] A. Cherkaev and L. Gibiansky (1993). Coupled estimates for the bulk and shear moduli of a two-dimensional isotropic elastic composite, *J. Mech. Phys. Solids*, **35**, 4391-4410.

[5] S. Czarnecki (2015). Isotropic Material Design, *Comp. Meth. Sci. Technol.*, **21**, 49-64.

[6] S. Czarnecki and T. Lewiński (2014). The Free Material Design in Linear Elasticity, In: G.I.N. Rozvany and T. Lewiński (Eds.) *Topology Optimization in Structural and Continuum Mechanics*, CISM International Centre for Mechanical Sciences (Udine), Springer, Wien, pp. 213-257.

[7] O. Sigmund (1994). Materials with prescribed constitutive parameters: An inverse homogenization problem, *Int. J. Solids Structures*, **17**, 2313-2329.