LAGRANGIAN FORMULATION OF HENCKY'S HYPERELASTIC MATERIAL MODEL: THEORY, EXPERIMENT, AND COMPUTER SIMULATION

***S.N. Korobeynikov*[1]*, A.Y. Larichkin*[1]*, T.A. Rotanova*[1] *and A.A. Oleinikov*[2]**
[1]*Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia*
[2]*Amur State University of Humanities and Pedagogy, Komsomolsk-na-Amure, Russia*

1. Introduction

Anand [1] has shown that Hencky's elasticity model describes the deformation of some isotropic hyperelastic materials at moderate strains (at a stretch of $0.7 \leq \lambda \leq 1.3$), including the deformation of vulcanized rubber and elastic deformations of metals under high pressures, i.e., this material model extends Hook's law from the range of infinitesimal strains to the range of large ones. In the present paper, we develop a new rate formulation of Hencky's hyperelastic isotropic material model. This material model is implemented in the MSC.Marc code. Experimental studies on torsion of free and constrained end rods of the Duothan QA965 polyurethane material have been performed. The deformation of the rods of this material is simulated using Hencky's elasticity model. The experimental data are compared with the results of computer simulations using the MSC.Marc code.

2. Rate formulation of Hencky's hyperelastic isotropic material model

Constitutive relations for Hencky's elasticity model in the Lagrangian formulation are [2]

(1) $$\bar{\boldsymbol{\tau}} = \mathbf{C}^E : \mathbf{E}^{(0)},$$

where $\mathbf{C}^E$ is the forth order tensor, $\mathbf{E}^{(0)} \equiv \sum_{i=1}^{m} \ln \lambda_i \mathbf{C}_i$ is right Hencky's (Lagrangian) strain tensor and $\bar{\boldsymbol{\tau}}$ is Noll's (Lagrangian) stress tensor [3], $\mathbf{C}_i$ are the eigenprojections [4] of the right (Lagrangian) Cauchy–Green strain tensor $\mathbf{C}$ [3], λ_i are the principal stretches [3], m is the m-index [4]; hereinafter, the indexes run over the values from 1 to m. The $\mathbf{C}^E$ tensor has the form [3]

(2) $$\mathbf{C}^E \equiv \lambda \mathbf{C}_{\text{I}} + 2\mu \mathbf{S}, \quad \mathbf{S} \equiv \frac{1}{2}(\mathbf{C}_{\text{II}} + \mathbf{C}_{\text{III}}) = \mathbf{I} \overset{\text{sym}}{\otimes} \mathbf{I}, \quad \mathbf{C}_{\text{I}} \equiv \mathbf{I} \otimes \mathbf{I}, \quad \mathbf{C}_{\text{II}} \equiv \mathbf{I} \underline{\otimes} \mathbf{I}, \quad \mathbf{C}_{\text{III}} \equiv \mathbf{I} \overline{\otimes} \mathbf{I}.$$

Hereinafter λ and μ are the Lamé parameters, $\mathbf{I}$ denotes the identity tensor, and $\otimes$, $\underline{\otimes}$, $\overline{\otimes}$, and $\overset{\text{sym}}{\otimes}$ denote the tensor product operations [5].

The Lagrangian rate form of relations (1) is written as

(3) $$\bar{\boldsymbol{\tau}}^D = \mathbf{C}^E : \mathbf{D},$$

where $\bar{\boldsymbol{\tau}}^D$ is Lagrangian objective corotational D-rate of the tensor $\bar{\boldsymbol{\tau}}$ introduced in [4, 6] (see also [7]), and $\mathbf{D}$ is the rotated strain rate tensor [3]. It can be shown that the Lagrangian rate formulation of Hencky's hyperelastic isotropic material model (3) is equivalent to the rate formulation

(4) $$\dot{\mathbf{S}}^{(2)} = \tilde{\mathbf{C}} : \dot{\mathbf{E}}^{(2)},$$

where the dot above a variable denotes the time derivative of this variable, $\mathbf{S}^{(2)}$ is the symmetric (Lagrangian) second Piola–Kirchhoff stress tensor, $\mathbf{E}^{(2)}$ is the symmetric (Lagrangian) Green–Lagrange strain tensor, and $\tilde{\mathbf{C}}$ is the Lagrangian objective fourth order elasticity tensor, which has two minor symmetries and one major symmetry [5]

(5) $$\tilde{\mathbf{C}} \equiv \sum_{i,j=1}^{m} \frac{\lambda}{\mu_i \mu_j} \mathbf{C}_i \otimes \mathbf{C}_j + \sum_{i=1}^{m} \frac{2(\mu - \tau_i)}{\mu_i^2} \mathbf{C}_i \overset{\text{sym}}{\otimes} \mathbf{C}_i + \sum_{i \neq j=1}^{m} \frac{2}{\mu_i - \mu_j} \left(\frac{\tau_i}{\mu_i} - \frac{\tau_j}{\mu_j} \right) \mathbf{C}_i \overset{\text{sym}}{\otimes} \mathbf{C}_j,$$

where $\mu_i = \lambda_i^2$ are the principal components of the right Cauchy–Green strain tensor $\mathbf{C}$ and τ_i are the principal components of Noll's stress tensor, which, for Hencky's hyperelastic isotropic material model, are defined as

$$(6) \quad \tau_i = \lambda \ln J + 2\mu \ln \lambda_i \quad (i = 1, \ldots, m), \quad J \equiv \lambda_1 \lambda_2 \lambda_3.$$

Using these equations, we can determine the second Piola–Kirchhoff stress tensor from the equation

$$(7) \quad \mathbf{S}^{(2)} = \sum_{i=1}^{m} \tau_i / \mu_i \mathbf{C}_i.$$

3. Experimental studies and computer simulation

The Lagrangian formulation of Hencky's isotropic hyperelastic material constitutive relations is implemented in the MSC.Marc code with user's subroutine hypela2.f. The reliability of the implementation is proved by comparison of the numerical solutions obtained using the MSC.Marc code with the exact solution of the two-dimensional problem of simple shear of a sample of Hencky's isotropic hyperelastic material. Experimental studies on extension of plane samples of the Duothan QA965 material have been performed. It has been shown that Hencky's isotropic hyperelastic material model describes the extension of samples of this material at a stretch of 1.5. Experiments and computer simulations of torsion of free and constrained end rods of the Duothan material have been performed. The experimental data have been found to be in satisfactory agreement with the exact solutions of these problems and with the results of computer simulations using the MSC.Marc code.

4. Conclusions

A new rate formulation of Hencky's hyperelastic isotropic material model is developed. Experimental studies on torsion of free and constrained end rods of the Duothan material were performed. In the theoretical and computer simulations of rod deformations, Hencky's isotropic hyperelastic material model was used. The experimental data are consistent with the exact solutions of the problems on rod torsion, as well as with the results of computer simulations using the MSC.Marc code.

5. References

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