OPTIMIZATION OF STRUCTURAL TOPOLOGY

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1. Introduction

The contemporary research towards optimization of structural topology comprises: a) topology optimization of trusses, b) two-material optimization relaxed by homogenization, c) free material design(FMD), see Bendsøe [1] and the review paper [2]. Michell trusses are exact solutions to problems (a) if the weight is minimized under the condition of the stress being bounded from both sides. The present paper presents the formulations and discusses the exemplary solutions of the following topology optimization methods referring to (a) and (c) class of problems:

i) Michell truss approach, see [3]
ii) Free material design (FMD), see [4]
iii) Cubic material design (CMD), see [5]
iv) Isotropic material design (IMD) , see [6,7]
v) Young modulus design (YMD)
vi) Design of thickness of in-plane loaded plates, see [8]

The main versions of problems (ii)-(vi) concern minimization of the compliance (if a single load case is discussed, n=1) or a convex combination of n compliances with the weights $\eta_\alpha,\ \alpha=1,...,n$ corresponding to subsequent loads applied non-simultaneously in case of n>1 load variants. In method (i) the areas of the cross sections may vanish. The methods (ii)-(v) involve the trace constraint (the integral of the trace of the Hooke tensor over the design domain is treated as given and equal to Λ) and refer to the settings with positive semi-definiteness condition. Thus the Kelvin moduli may vanish in some subdomains of the design domain Ω. In method (vi) the thickness may be zero thus enabling making holes in the plate. Just this feature- admitting zero values of the design variables- distinguishes the topology optimization from size optimization. It turns out that all the problems (i)-(vi) may be naturally reduced to the two mutually dual problems similar to those appearing in the theory of optimal transportation of Monge-Kantorovich. They will be called: *governing problems* and they unify all the topology optimization methods (i-vi).

2. **The governing problems**

Let $\Sigma_\alpha(\Omega)$ be the set of statically admissible trial stresses corresponding to α th load variant, $\alpha=1,...,n$. The *governing problem* in its *primal* setting has always the form:

$$(1)\qquad Z=\min_{\boldsymbol{\tau}^{(\alpha)}\in\Sigma_\alpha(\Omega)}\int_\Omega \left|\left|\left|\sqrt{\eta_1}\boldsymbol{\tau}^{(1)},...,\sqrt{\eta_n}\boldsymbol{\tau}^{(n)}\right|\right|\right| dx,$$

where the integrand is a certain norm of the collection of trial stresses equilibrating subsequent load variants. The *governing problem dual* to (1) reads

$$(2)\quad Z=\max\left\{\sum_{\alpha=1}^{n}\sqrt{\eta_\alpha}\, f^{(\alpha)}\left(\mathbf{v}^{(\alpha)}\right)\middle|\ \mathbf{v}^{(\alpha)}\ \text{kin.admiss.};\ \left|\left|\left|\boldsymbol{\varepsilon}^{(1)},...,\boldsymbol{\varepsilon}^{(n)}\right|\right|\right|^{*}\le 1\ \text{ a.e. in } \Omega,\ \text{with }\ \boldsymbol{\varepsilon}^{(\alpha)}=\boldsymbol{\varepsilon}(\mathbf{v}^{(\alpha)})\right\}$$

in which the linear forms $f^{(\alpha)}(.)$ correspond to virtual works of the loads, $\boldsymbol{\varepsilon}(.)$ is the symmetric part of the gradient and the dual norm involved is defined by

(3) $$\left\|\left|\boldsymbol{\varepsilon}^{(1)},\ldots,\boldsymbol{\varepsilon}^{(n)}\right\|\right|^{*}=\max_{\substack{\boldsymbol{\sigma}^{(\alpha)}\neq\mathbf{0},\\ \alpha=1,\ldots,n}}\left(\sum_{\alpha=1}^{n}\boldsymbol{\varepsilon}^{(\alpha)}\cdot\boldsymbol{\sigma}^{(\alpha)}\Big/\left\|\left|\boldsymbol{\sigma}^{(1)},\ldots,\boldsymbol{\sigma}^{(n)}\right\|\right|\right)\ .$$

The formulae (1-3) apply to all the methods (i-vi), see Refs.[2-8] for the definitions of the norm involved in (1). The theory of YMD is still unpublished; the next section delivers some details.

3. Young modulus design. Final remarks

The distribution of the Poisson ratio ν within the design domain is treated as prescribed, while the distribution of the Young modulus E is subject to optimization. The trace of the Hooke tensor, expressed in terms of these moduli, represents the unit cost. The governing equation (1) holds with the norm(for 3D case)

(4) $$\left\|\left|\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)},\ldots,\boldsymbol{\sigma}^{(n)}\right\|\right|=\sqrt{\frac{6-9\nu}{1+\nu}\sum_{\alpha=1}^{n}\left(\frac{\operatorname{tr}\boldsymbol{\sigma}^{(\alpha)}}{\sqrt{3}}\right)^{2}+\frac{6-9\nu}{1-2\nu}\sum_{\alpha=1}^{n}\left\|\operatorname{dev}\boldsymbol{\sigma}^{(\alpha)}\right\|^{2}}\ .$$

Let $\breve{\boldsymbol{\tau}}^{(\alpha)}$ be minimizers of (1) and $\breve{\boldsymbol{\sigma}}^{(\alpha)}=\sqrt{\eta_\alpha}\,\breve{\boldsymbol{\tau}}^{(\alpha)}$. The optimal Young modulus equals $\breve{E}=\breve{E}_1/a$ with $a=(6-9\nu)\big/\big((1+\nu)(1-2\nu)\big)$ and

(5) $$\breve{E}_1=\frac{\Lambda}{\int_\Omega\sqrt{\left\|\left|\breve{\boldsymbol{\sigma}}^{(1)},\ldots,\breve{\boldsymbol{\sigma}}^{(n)}\right\|\right|}\,\mathrm{d}x}\left\|\left|\breve{\boldsymbol{\sigma}}^{(1)},\ldots,\breve{\boldsymbol{\sigma}}^{(n)}\right\|\right|$$

The numerical method proposed in Ref.[6] applies to solving (1,4). Due to the governing problems (1) and (2) being similar for all the methods (i-vi) one can expect strong similarities between the final optimal layouts corresponding to these problems having different original formulations.

The methods (ii)-(v) deliver the numerical algorithms for programming the additive fabrication processes.

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6. References

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